C. R. Acad. Sci. Paris, t. 332, Série I, p. 45–50, 2001
Géométrie différentielle/*Differential Geometry*

The Euler class for Riemannian flows

José Ignacio ROYO PRIETO

Departamento de Matemáticas, Universidad del País Vasco-Euskal Herriko Unibertsitatea, Apdo. 644, 48080 Bilbao, Spain
E-mail: mtbroprj@lg.ehu.es

(Reçu le 25 octobre 2000, accepté le 6 novembre 2000)

Abstract. Let $\mathcal{F}$ be a Riemannian flow on a closed manifold M. The Gysin sequence relating the basic cohomology of $\mathcal{F}$ and the de Rham cohomology of M has been constructed for isometric flows in [6]. In this paper, we extend it to the case of Riemannian flows. We also give a geometric characterization of the vanishing of the Euler class similar to the one given in [9]. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

La classe d'Euler pour les flots riemanniens

Résumé. *Soit $\mathcal{F}$ un flot riemannien sur une variété fermée M. La suite de Gysin, qui relie la cohomologie basique de $\mathcal{F}$ et la cohomologie de de Rham de M, est construite pour les flots isométriques dans* [6]. *Dans cette Note, on l'étend au cas des flots riemanniens. On donne une caractérisation géométrique de la nullité de la classe d'Euler, semblable à celle donnée dans* [9]. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit M une variété lisse fermée. Une action de $\mathbb{R}$ sans points fixes définit un feuilletage $\mathcal{F}$ de dimension 1. Nous notons $\Omega^*(M/\mathcal{F})$ le complexe des formes basiques et $\mathrm{H}^*(M/\mathcal{F})$ sa cohomologie. Nous sommes intéressés dans la relation entre la cohomologie de de Rham de M et la cohomologie basique de $\mathcal{F}$. Quand l'action est périodique nous avons en fait une action du cercle $\mathbb{S}^1$, et cette relation est décrite par la *suite de Gysin* classique. La suite de Gysin a été construite dans [6] pour des flots isométriques, c'est-à-dire des actions de $\mathbb{R}$ qui préservent une métrique riemannienne μ de M. La suite de Gysin obtenue est :

$$\cdots \longrightarrow \mathrm{H}^i(M/\mathcal{F}) \longrightarrow \mathrm{H}^i(M) \longrightarrow \mathrm{H}^{i-1}(M/\mathcal{F}) \stackrel{\wedge[\mathrm{d}\chi]}{\longrightarrow} \mathrm{H}^{i+1}(M/\mathcal{F}) \longrightarrow \cdots,$$

où le morphisme de connexion est la multiplication par la classe d'Euler $[\mathrm{d}\chi] \in \mathrm{H}^2(M/\mathcal{F})$, avec χ la forme caractéristique du flot. Le fait que l'adhérence de $\mathbb{R}$ dans le groupe de difféomorphismes de M est compact est crucial dans la démonstration.

Dans cette Note, nous étendons ce résultat au cas des flots riemanniens. Un flot est riemannien quand il existe sur M une métrique riemannienne μ dont la composante orthogonale est invariante par le flot.

Note présentée par Étienne GHYS.

S0764-4442(00)01770-5/FLA
© 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés.

Une telle métrique est dite *quasi-fibrée*. Nous avons deux différences importantes par rapport au cas isométrique : d'une part, la forme $\mathrm{d}\chi$ n'est pas toujours basique, et d'autre part, l'adhérence de $\mathbb{R}$ dans le groupe de difféomorphismes de M n'est pas forcément compacte. C'est pour cela que nous utiliserons une approche différente, en utilisant fortement la géométrie locale de $\mathcal{F}$, décrite par Carrière en [4].

Comme indiqué, nous avons la décomposition $\mathrm{d}\chi = \mathfrak{e} + \chi \wedge \kappa$, où $\mathfrak{e}$ est la *forme d'Euler* et κ la forme de courbure moyenne. Domínguez a prouvé dans [5] qu'il existe toujours une métrique riemannienne μ telle que κ soit basique. Dans ces conditions, κ est fermée, et la classe $[\kappa] \in \mathrm{H}^1(M/\mathcal{F})$ ne dépend pas de la métrique μ choisie (*voir* [2]). La cohomologie basique est la cohomologie duale de la cohomologie tordue $\mathrm{H}^*_\kappa(M/\mathcal{F})$, qui est définie à partir du complexe basique muni de la différentielle $\mathrm{d}_\kappa\omega = \mathrm{d}\omega - \kappa \wedge \omega$ (*voir* [7]). Nous utilisons aussi la différentielle $\mathrm{d}_{-\kappa}\omega = \mathrm{d}\omega + \kappa \wedge \omega$. La forme d'Euler est basique, et vérifie $\mathrm{d}_{-\kappa}\mathfrak{e} = 0$.

En utilisant la description de la structure locale décrite dans la proposition 1 et un argument à la Mayer–Vietoris (*voir* [3], p. 289), nous prouvons le quasi-isomorphisme :

$$\big(\Omega^*(M), \mathrm{d}\big) \simeq \big(\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D\big),$$

où la différentielle D est donnée par la formule $D(\alpha, \beta) = (\mathrm{d}\alpha + \mathfrak{e} \wedge \beta, -\mathrm{d}\beta + \kappa \wedge \beta)$. À partir de ce quasi-isomorphisme, nous trouvons directement la suite de Gysin.

THÉORÈME 1 (Suite de Gysin). – *Soit $\mathcal{F}$ un flot riemannien sur une variété lisse fermée M, et est μ une métrique quasi-fibrée dont la courbure moyenne κ soit basique. Alors, nous avons la suite exacte longue* :

$$\cdots \longrightarrow \mathrm{H}^i(M/\mathcal{F}) \longrightarrow \mathrm{H}^i(M) \longrightarrow \mathrm{H}^{i-1}_\kappa(M/\mathcal{F}) \xrightarrow{\wedge[\mathfrak{e}]} \mathrm{H}^{i+1}(M/\mathcal{F}) \longrightarrow \cdots,$$

où le morphisme de connexion est, au signe près, la multiplication par la classe d'Euler $[\mathfrak{e}] \in \mathrm{H}^2_{-\kappa}(M/\mathcal{F})$.

La suite de Gysin obtenue ne dépend pas de la métrique choisie.

La forme d'Euler $\mathfrak{e}$ s'annule si et seulement si le supplémentaire orthogonal de $\mathcal{F}$ est intégrable. Ces conditions ont été affaiblies dans [9] pour un flot isométrique. Dans ce cas, la classe d'Euler $[\mathrm{d}\chi] \in \mathrm{H}^2(M/\mathcal{F})$ est un invariant du feuilletage, et sa nullité équivaut à l'existence d'une fibration transverse à $\mathcal{F}$. Pour un flot riemannien, nous prouvons que la classe d'Euler $[\mathfrak{e}] \in \mathrm{H}^2_{-\kappa}(M/\mathcal{F})$ est indépendante de la métrique, dans le sens de la proposition 3. En particulier, la nullité de la classe d'Euler est un invariant du feuilletage, et cette nullité est équivalente à l'existence d'un feuilletage transverse $\mathcal{G}$ avec une condition supplémentaire. Nous dirons que la torsion d'un feuilletage $\mathcal{G}$ transverse à $\mathcal{F}$, relativement à un champ non singulier X définissant $\mathcal{F}$, est basique si $L_X L_X \omega = 0$, où ω est la 1-forme qui définit $\mathcal{G}$ normalisée par $\omega(X) = 1$. Le théorème qui caractérise la nullité de la classe d'Euler est :

THÉORÈME 2. – *Pour un flot riemannien $\mathcal{F}$ sur une variété lisse M, les deux conditions suivantes sont équivalentes* :
(i) *la classe d'Euler s'annule* ;
(ii) *il existe un feuilletage $\mathcal{G}$ transverse à $\mathcal{F}$ dont la torsion est basique.*

1. Introduction

Let M be a closed $(n+1)$-manifold. An action $\Psi : \mathbb{R} \times M \to M$ without fixed points defines a 1-dimensional foliation $\mathcal{F}$. We are interested in the relation between the de Rham cohomology of M and the basic cohomology of $\mathcal{F}$, which is the natural cohomology for the study of the orbit space of Ψ. When the action is periodic, we have indeed a circle action, and this relation is described by the classical

Gysin sequence. In [6], the Gysin sequence has also been constructed for an *isometric flow*, i.e., an action $\Psi : \mathbb{R} \times (M, \mu) \to (M, \mu)$ preserving a Riemannian metric μ on M. We denote by $\Omega^*(M/\mathcal{F})$ the complex of *basic forms*, and by $\mathrm{H}^*(M/\mathcal{F})$ its cohomology (*basic cohomology*). The Gysin sequence we have is:

$$\cdots \longrightarrow \mathrm{H}^i(M/\mathcal{F}) \longrightarrow \mathrm{H}^i(M) \longrightarrow \mathrm{H}^{i-1}(M/\mathcal{F}) \stackrel{\wedge[\mathrm{d}\chi]}{\longrightarrow} \mathrm{H}^{i+1}(M/\mathcal{F}) \longrightarrow \cdots, \tag{1}$$

where the connecting morphism is multiplication by the Euler class $[\mathrm{d}\chi] \in \mathrm{H}^2(M/\mathcal{F})$, being χ the characteristic form of the flow. The tool used to get the Gysin sequence is the quasi-isomorphism (i.e., an isomorphism in cohomology):

$$\big(\Omega^*(M), \mathrm{d}\big) \simeq \big(\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D\big), \tag{2}$$

where d is the exterior differential and D is the twisted differential $D(\alpha, \beta) = (\mathrm{d}\alpha + \mathrm{d}\chi \wedge \beta, -\mathrm{d}\beta)$. It is crucial that the closure of the 1-parameter group $\{\Psi_t\}_t$ in the group of diffeomorphisms of M is compact.

In this paper we extend the scope of the Gysin sequence to Riemannian flows. A flow $\mathcal{F}$ on M is *Riemannian* if there exists a Riemannian metric μ whose orthogonal component is invariant by $\mathcal{F}$. Such a metric is said to be *bundle-like*. Note that when $\mathcal{F}$ is isometric, the entire metric must be invariant. The case of a Riemannian flow has two important differences with the isometric one. In one hand, $\mathrm{d}\chi$ is basic for an isometric flow, but in the Riemannian case, we have a decomposition $\mathrm{d}\chi = \mathfrak{e} + \chi \wedge \kappa$, where $\mathfrak{e}$ and κ are respectively the Euler form and the mean curvature form of the flow, which can be supposed to be basic for a suitable bundle-like metric μ (*see* [5]). The second main difference is that in the Riemannian case, the closure of $\{\Psi_t\}_t$ is no longer necessarily compact. So, we need a different approach to get a decomposition like (2). The Gysin sequence we obtain is:

$$\cdots \longrightarrow \mathrm{H}^i(M/\mathcal{F}) \longrightarrow \mathrm{H}^i(M) \longrightarrow \mathrm{H}^{i-1}_\kappa(M/\mathcal{F}) \stackrel{\wedge[\mathfrak{e}]}{\longrightarrow} \mathrm{H}^{i+1}(M/\mathcal{F}) \longrightarrow \cdots,$$

where $\mathrm{H}^*_\kappa(M/\mathcal{F})$ denotes the κ-twisted cohomology, and the connecting morphism $\wedge[\mathfrak{e}]$ is, up to sign, multiplication by the Euler class $[\mathfrak{e}] \in \mathrm{H}^2_{-\kappa}(M/\mathcal{F})$. The Gysin sequence above is derived canonically from the quasi-isomorphism:

$$\big(\Omega^*(M), \mathrm{d}\big) \simeq \big(\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D\big),$$

where the differential D is defined by the formula $D(\alpha, \beta) = (\mathrm{d}\alpha + \mathfrak{e} \wedge \beta, -\mathrm{d}\beta + \kappa \wedge \beta)$. To get this result, we use strongly the local structure of $\mathcal{F}$. Finally, we give a geometric characterization of the vanishing of the Euler class similar to the one given in [9] in the isometric case. The vanishing of the Euler class is equivalent to the presence of a foliation transverse to the flow, with an additional condition, stated in Theorem 2.

2. Riemannian flows and local structure

Recall that a flow is a tangentially oriented 1-dimensional foliation $\mathcal{F}$. The flow $\mathcal{F}$ is *Riemannian* if there exists a holonomy invariant metric μ. We can choose a nonsingular smooth vector field X defining $\mathcal{F}$ and satisfying $\mu(X, X) = 1$. The *characteristic form* is the 1-form $\chi = i_X \mu$, where i_X denotes the contraction by X. The *mean curvature form* is the 1-form defined by $\kappa = L_X \chi$, where L_X denotes the Lie derivative respect to X. This two forms depend on the flow $\mathcal{F}$ and the metric μ. A form $\omega \in \Omega^*(M)$ is said to be *basic* if for every vector field V tangent to $\mathcal{F}$, we have $i_V \omega = i_V \mathrm{d}\omega = 0$. We denote by $\Omega^*(M/\mathcal{F})$ the complex of the basic forms, and its cohomology by $\mathrm{H}^*(M/\mathcal{F})$.

It has been shown in [5] that for a Riemannian flow $\mathcal{F}$, we can choose a bundle-like metric μ such that its mean curvature form κ is basic. In these conditions, κ is closed, and the class $[\kappa] \in \mathrm{H}^1(M/\mathcal{F})$ is an invariant of the foliation that we shall call the *Álvarez class* (*see* [2]). We have the decomposition

$$\mathrm{d}\chi = \mathfrak{e} + \chi \wedge \kappa.$$

The *Euler form* $\mathfrak{e}$ thus defined, is basic and satisfies $\mathrm{d}\mathfrak{e} = -\mathfrak{e} \wedge \kappa$.

Let's denote by $\mathrm{H}^*_\kappa(M/\mathcal{F})$ the *κ-twisted cohomology*, which is the cohomology of the complex of basic forms endowed with the differential $\mathrm{d}_\kappa \omega = \omega - \kappa \wedge \omega$. This cohomology depends only on the basic class of κ. The basic cohomology does not satisfy Poincaré duality for an arbitrary foliation, but for a Riemannian flow, it is dual to the κ-twisted cohomology, i.e., $\mathrm{H}^i(M/\mathcal{F}) \cong \mathrm{H}^{n-i}_\kappa(M/\mathcal{F})$ for every $i \geqslant 0$ (*see* [7]). We also define the $(-\kappa)$-*twisted differential* by $\mathrm{d}_{-\kappa}\omega = \mathrm{d}\omega + \kappa \wedge \omega$.

The local structure of $\mathcal{F}$ has been described in [4]. If we denote by L a leaf of $\mathcal{F}$, we have:

PROPOSITION 1 (Carrière). – *There exists a $\mathcal{F}$-saturated neighbourhood $U \subseteq M$ of $\overline{L}$ such that*:

(i) *there is a diffeomorphism $U \to \mathbb{S}^1 \times \mathbb{T}^k \times \mathbb{D}^{n-k}$ mapping $\overline{L}$ onto $\mathbb{S}^1 \times \mathbb{T}^k \times \{0\}$;*

(ii) *the flow $\mathcal{F}$ restricted to U is conjugated to the flow obtained by the suspension of a diffeomorphism γ of $\mathbb{T}^k \times \mathbb{D}^{n-k}$ given by the formula $\gamma(x,y) = (T(x), R(y))$, where T is an irrational translation of $\mathbb{T}^k$ and R is a rotation of $\mathbb{R}^{n-k}$.*

We shall refer to such a neighbourhood as a *Carrière neighbourhood*.

3. Gysin sequence

The *short Gysin sequence* is the following sequence of differential complexes:

$$0 \longrightarrow \Omega^*(M/\mathcal{F}) \xrightarrow{\iota} \Omega^*(M) \xrightarrow{\rho} \Omega^*(M)/\Omega^*(M/\mathcal{F}) \longrightarrow 0, \tag{3}$$

where ρ is the projection induced by the inclusion ι. The cohomology of $\Omega^*(M)/\Omega^*(M/\mathcal{F})$ is called the *$\mathcal{F}$-relative de Rham cohomology*, and some of its properties have been studied in [8]. Our aim is to describe it in terms of basic forms. To achieve this, we use the following:

PROPOSITION 2. – *In the above conditions, there is a quasi-isomorphism*:

$$\begin{aligned} \Phi : \big(\Omega^*(M/\mathcal{F}) \oplus \Omega^*(M/\mathcal{F}), D\big) &\longrightarrow \big(\Omega^*(M), \mathrm{d}\big), \\ (\alpha, \beta) &\longmapsto \alpha + \chi \wedge \beta, \end{aligned} \tag{4}$$

where the differential D is defined by $D(\alpha,\beta) = (\mathrm{d}\alpha + \mathfrak{e} \wedge \beta, -\mathrm{d}\beta + \kappa \wedge \beta)$.

Proof. – Consider on M the foliation $\overline{\mathcal{F}}$ defined by the closures of the leaves of $\mathcal{F}$. Applying a Mayer–Vietoris argument ([3], p. 289) to the leaf space $M/\overline{\mathcal{F}}$, which is a stratified manifold, and using induction on its depth, it suffices to prove the theorem for the foliation $\mathcal{F}_U$ induced on an arbitrary Carrière neighbourhood U. Let's denote by Φ_U, D_U, μ_U, κ_U and $\mathfrak{e}_U$ the correspondent restrictions of Φ, D, μ, κ and $\mathfrak{e}$. Since with the flat metric on U the mean curvature form vanishes, there exists a basic function f such that $\kappa = \mathrm{d}f$. With the change of metric $\mu'_U = \mathrm{e}^{2f}\mu_U$, we have $X' = \mathrm{e}^{-f}X$, $\chi' = \mathrm{e}^f\chi$, $\kappa'_U = 0$ and $\mathfrak{e}'_U = \mathrm{e}^f\mathfrak{e}_U$. We construct the following differential applications:

$$\big(\Omega^*(U/\mathcal{F}_U) \oplus \Omega^{*-1}(U/\mathcal{F}_U), D_U\big) \xrightarrow{\varepsilon} \big(\Omega^*(U/\mathcal{F}_U) \oplus \Omega^{*-1}(U/\mathcal{F}_U), D'_U\big) \xrightarrow{\Phi'_U} \big(\Omega^*(U), \mathrm{d}\big),$$

where $\varepsilon(\alpha,\beta) = (\alpha, \mathrm{e}^{-f}\beta)$, $D'_U(\alpha,\beta) = (\mathrm{d}\alpha + \mathfrak{e}'_U \wedge \beta, -\mathrm{d}\beta)$ and $\Phi'_U(\alpha,\beta) = \alpha + \chi' \wedge \beta$. Note that X' is a Killing vector field (*see* [10]), and thus Φ'_U is a quasi-isomorphism. One can check directly that ε is an isomorphism, and as $\Phi'_U \circ \varepsilon = \Phi_U$, the proof is complete. □

From (5) and (7), we get a quasi-isomorphism $(\Omega^*(M/\mathcal{F}), \mathrm{d}_\kappa) \xrightarrow{\Phi_\kappa} (\Omega^*(M)/\Omega^*(M/\mathcal{F}), \mathrm{d})$ given by $\Phi_\kappa(\omega) = \overline{\chi \wedge \omega}$. A direct consequence is the following:

THEOREM 1 (Gysin Sequence). – *Let $\mathcal{F}$ be a Riemannian flow on a closed manifold M, and choose a metric μ such that the mean curvature form κ is basic. Then, we have the following long exact sequence*:

$$\cdots \longrightarrow \mathrm{H}^i(M/\mathcal{F}) \xrightarrow{\iota^*} \mathrm{H}^i(M) \xrightarrow{(\Phi_\kappa^*)^{-1}\circ\rho^*} \mathrm{H}_\kappa^{i-1}(M/\mathcal{F}) \xrightarrow{\wedge[\mathfrak{e}]} \mathrm{H}^{i+1}(M/\mathcal{F}) \longrightarrow \cdots, \tag{5}$$

where the connecting morphism is, up to sign, multiplication by the Euler class $[\mathfrak{e}] \in \mathrm{H}^2_{-\kappa}(M/\mathcal{F})$.

The sequence above is called the *long Gysin sequence*. In the case of an isometric flow, we can choose a metric such that the mean curvature form vanishes. So, (5) generalizes (1).

Remark 1. – The long Gysin sequence does not depend on the metric μ.

Remark 2. – Combining the quasi-isomorphism Φ_κ and the duality result of [7], we obtain that the terms $E_2^{\cdot,1}$ and $E_2^{\cdot,0}$ in the spectral sequence of a Riemannian flow are dual. This has been proven with more generality in [1] using analytic techniques, different from our geometric approach.

4. Vanishing of the Euler class

The Euler form $\mathfrak{e}$ vanishes if and only if the orthogonal complement of $\mathcal{F}$ is involutive. These equivalent conditions have been weakened in [9] for an isometric flow. In that case, the Euler class $[\mathrm{d}\chi] \in \mathrm{H}^2(M/\mathcal{F})$ is an invariant of the foliation, and its vanishing is equivalent to the presence of a fibration transverse to $\mathcal{F}$. For a Riemannian flow, we prove that the vanishing of the Euler class is equivalent to the existence of a foliation supplementary to $\mathcal{F}$ with an extra condition. Notice first that the Euler class $[\mathfrak{e}] \in \mathrm{H}^2_{-\kappa}(M/\mathcal{F})$ does not depend on the metric in the sense of the following:

PROPOSITION 3. – *Let μ_1 and μ_2 be two bundle-like metrics with basic mean curvature forms κ_1 and κ_2. Consider the canonical* (*up to a multiplicative positive constant*) *isomorphism*

$$\varphi : \mathrm{H}^2_{-\kappa_2}(M/\mathcal{F}) \longrightarrow \mathrm{H}^2_{-\kappa_1}(M/\mathcal{F}), \tag{6}$$

given by $\varphi([\omega]) = [\mathrm{e}^f \omega]$, where $\mathrm{d}f = \kappa_2 - \kappa_1$. Then, $\varphi[\mathfrak{e}_2]$ and $[\mathfrak{e}_1]$ are proportional. In particular, the vanishing of the Euler class does not depend on μ, but just on $\mathcal{F}$.

Proof. – With the same techniques used in Proposition 2, we can construct a quasi-isomorphism $(\Omega^*(M/\mathcal{F}) \oplus \Omega^{*-1}(M/\mathcal{F}), D^\circ) \simeq (\Omega^*(M), \mathrm{d}_{-\kappa})$, with differential $D^\circ(\alpha,\beta) = (\mathrm{d}_{-\kappa}\alpha + \mathfrak{e} \wedge \beta, -\mathrm{d}\beta)$, for every basic mean-curvature form κ. This leads us to the *twisted Gysin sequence*:

$$\cdots \longrightarrow \mathrm{H}^i_{-\kappa}(M/\mathcal{F}) \longrightarrow \mathrm{H}^i_{-\kappa}(M) \longrightarrow \mathrm{H}^{i-1}(M/\mathcal{F}) \xrightarrow{\wedge[\mathfrak{e}]} \mathrm{H}^{i+1}_{-\kappa}(M/\mathcal{F}) \longrightarrow \cdots.$$

The isomorphism (6) induces a chain morphism between the twisted Gysin sequences for μ_1 and μ_2, yielding the result. □

Let $\mathcal{G}$ be a foliation transverse to a nonsingular vector field X defining $\mathcal{F}$. There is a unique 1-form ω determined by $i_X\omega = 1$ and $\ker\omega = T\mathcal{G}$. We mean by a *torsion* of $\mathcal{G}$ respect to X a 1-form τ such that $\mathrm{d}\omega = \omega \wedge \tau$. Note that if a torsion is basic for $\mathcal{F}$, then it must be $L_X\omega$.

THEOREM 2. – *For a Riemannian flow $\mathcal{F}$, the following conditions are equivalent*:

(i) *the Euler class vanishes*;

(ii) *there exists a foliation $\mathcal{G}$ transverse to $\mathcal{F}$, with basic torsion.*

Proof. – Assume (i), and take $\gamma \in \Omega^1(M/\mathcal{F})$ such that $\mathrm{d}_{-\kappa}\gamma = \mathfrak{e}$. Then, $\omega = \chi - \gamma$ is integrable, defining a foliation $\mathcal{G}$ that satisfies (ii). For the converse, pick a bundle-like metric μ and construct $\nu = \omega \otimes \omega + \mu_{\mathcal{G}}$. Then, ν is bundle-like, $\kappa_\nu = L_X\omega$ is basic, and $\mathfrak{e}_\nu = 0$. □

Remark 3. – For a Riemannian flow, the Álvarez class and the Euler class are independent.

Remark 4. – The vanishing of the Euler class of $\mathcal{F}$ implies the vanishing of the Godbillon–Vey class of the transverse foliation $\mathcal{G}$ described in Theorem 2.

Acknowledgements. This research has been supported by Gobierno Vasco-Eusko Jaurlaritza and UPV 127.310-EA 005/99. I wish to thank Professeur Saralegi for introducing me to this research subject and for his helpful discussions and suggestions.

References

[1] Álvarez J.A., Duality in the spectral sequence of Riemannian foliations, Amer. J. Math. 111 (1989) 905–925.
[2] Álvarez J.A., The basic component of the mean curvature form of Riemannian foliations, Ann. Global Anal. and Geom. 10 (1992) 179–194.
[3] Bredon G.E., Topology and Geometry, Springer-Verlag, 1993.
[4] Carrière Y., Flots riemanniens, in: Astérisque 116, 1984, pp. 31–52.
[5] Domínguez D., Finiteness and tenseness theorems for Riemannian foliations, Amer. J. Math. 120 (1998) 1237–1276.
[6] Kamber F., Tondeur Ph., Foliations and metrics, in: Proc. of a Year in Differential Geometry, University of Maryland, Birkhäuser Progr. Math. 32, 1983, pp. 103–152.
[7] Kamber F., Tondeur Ph., Duality theorems for foliations, Astérisque 16 (1984) 458–471.
[8] Rummler H., Quelques notions simples en géometrie riemannienne et leurs applications aux feuilletages compacts, Comment. Math. Helv. 54 (1979) 224–239.
[9] Saralegi M., The Euler class for flows of isometries, in: Pitman Research Notes in Math. 131, 1985, pp. 220–227.
[10] Tondeur Ph., A characterization of Riemannian flows, Proc. Amer. Math. Soc. 125 (11) (1997) 3403–3405.